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Conformal invariance and critical dynamics

John L Cardy

Department of Physics, University of California, Santa Barbara, California 93106, USA

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Abstract. We discuss the behaviour of dynamic correlation functions under conformal transformations. In particular we obtain the exact form of the dynamic scaling function at the critical point for an arbitrary isotropic two-dimensional model, with either conserved or non-conserved order parameter.

1. Introduction

The principle of conformal invariance of isotropic systems at a critical point has been shown to have striking consequences, particularly in two dimensions (e.g. see Polyakov 1970, Belavin *et al* 1984, Dotsenko 1984, Cardy 1984, Cardy and Redner 1984). It rests on the idea that correlation functions at the critical point should transform simply not only under a uniform global length rescaling $\boldsymbol{r}' = b^{-1}\boldsymbol{r}$, but also under more general conformal transformations, which correspond locally to a rotation plus a dilation by an r-dependent rescaling factor $b(\boldsymbol{r})$, In two dimensions any analytic function $w = f(\zeta)$ gives a conformal transformation. In general such a transformation will distort the boundary conditions. Sometimes this feature can be exploited to gain useful information. For example, the transformation $w = \ln \zeta$ maps the whole ζ plane into a strip, or cylinder, with periodic boundary conditions, which is a quasi-one-dimensional geometry. This then gives a relation between 1D systems and 2D critical systems (Cardy 1984), which will be further exploited in this paper.

So far, conformal invariance has been applied only to static correlation functions. It is the purpose of this paper to extend the discussion to critical dynamics. According to the theory of dynamic scaling (see, e.g., Ma 1976) (which is supported by numerous renormalisation group calculations) the dynamic response function $\tilde{G}(\omega, \boldsymbol{k})$ at the critical point scales according to

$$\tilde{G}(\omega, \boldsymbol{k}) \sim A k^{-2+\eta} \Phi(B\omega k^{-z}) \tag{1.1}$$

in the scaling limit $k \to 0$, ωk^{-z} fixed, where z is a universal dynamic exponent, Φ is a universal scaling function, and A, B are non-universal constants. The main result of this paper is that, in two dimensions, conformal invariance completely determines the scaling function Φ. The argument proceeds in two stages. First, in § 2, we discuss the properties of the dynamic correlation function under a general conformal transformation. It turns out that the usual static transformation law is augmented by a rescaling of the microscopic rate $\Gamma \to b(\boldsymbol{r})^{-z}\Gamma$. Under the logarithmic transformation, the 2D critical dynamics is then mapped onto the dynamics of a strip with a *non-uniform* rate. The static correlation length of the system in the strip is of the same order of magnitude as its width. Therefore, on much larger distance scales it is permissible to use mean field theory to calculate the dynamic correlation function in the strip. This is tractable even for a non-uniform rate. In this paper we solve for the dynamic correlation function for two simple types of dynamics: model A (corresponding to spin-flip dynamics (Glauber 1963)) and model B (corresponding to a converved order parameter or spin-exchange dynamics (Kawasaki 1970)). The relevant equations are formulated and solved in § 3. Finally, we discuss the physical interpretation of our results in § 4.

2. Conformal invariance and dynamics

The time-dependent correlation function in the disordered phase is

$$C(|\boldsymbol{r}_1-\boldsymbol{r}_2|, t_1-t_2)=\langle\varphi(\boldsymbol{r}_1, t_1)\varphi(\boldsymbol{r}_2, t_2)\rangle \tag{2.1}$$

where $\varphi(\boldsymbol{r}, t)$ is the instantaneous value of the order parameter. $C(r, t)$ depends also on thermodynamic variables such as temperature and applied field. At a critical point, the theory of dynamic scaling (Halperin and Hohenberg 1969) asserts that for large r, C transforms according to

$$C(r, t)=b^{-(d-2+\eta)}C(b^{-1}r, b^{-z}t) \tag{2.2}$$

under a uniform dilatation $\boldsymbol{r}\to b^{-1}\boldsymbol{r}$. This covariance is simply interpreted in the language of the renormalisation group (Ma 1976). According to these ideas, such a dilatation is equivalent to a coarse-graining procedure in which the microscopic length scale a is *increased* by a uniform factor b. At a critical point, the form of the Hamiltonian (which governs the static properties) is invariant under this coarse graining. The coarse-grained local order parameter is simply related to the original one by a rescaling $\varphi\to b^{-x}\varphi$ where $2x=d-2+\eta$. Thus the static correlation function satisfies

$$\langle\varphi(\boldsymbol{r}_1)\varphi(\boldsymbol{r}_2)\rangle=b^{-x}b^{-x}\langle\varphi(\boldsymbol{r}_1')\varphi(\boldsymbol{r}_2')\rangle \tag{2.3}$$

where $\boldsymbol{r}_i'=b^{-1}\boldsymbol{r}_i$, which is equivalent to (2.2) when $t_1=t_2$.

In order to incorporate the dynamics, one brings in the idea that the local microscopic rate Γ transforms under this coarse graining according to $\Gamma\to b^{-z}\Gamma$. Since the correlation function depends on Γ and t only through the combination Γt, this is equivalent to (2.2).

The main assumption of the RG framework is that of locality: the fixed point Hamiltonian is short ranged, and the coarse-grained order parameter φ and microscopic rate Γ depend only locally on their RG ancestors. This allows the generalisation of the concept of scale invariance to that of conformal invariance. A conformal transformation $\boldsymbol{r}\to\boldsymbol{r}'$ is one which is *locally* equivalent to a dilatation by a factor $b(\boldsymbol{r})$, plus a possible rotation. That is, there is no shear. Under such a transformation, the fixed point Hamiltonian will remain invariant, and locally φ and Γ will transform according to

$$\varphi(\boldsymbol{r})\to b(\boldsymbol{r})^{-x}\varphi(\boldsymbol{r}') \qquad \Gamma\to b(\boldsymbol{r})^{-z}\Gamma. \tag{2.4}$$

The second equation implies that, under a general conformal transformation, the system will be transformed into one with a *non-uniform* rate $\Gamma(\boldsymbol{r})$. It is necessary, then, to consider this possibility from the outset, and regard the correlation function $C(\boldsymbol{r}_1, \boldsymbol{r}_2; \Gamma(\boldsymbol{r})t)$ as a functional of $\Gamma(\boldsymbol{r})$. The transformation law is then

$$C(\boldsymbol{r}_1, \boldsymbol{r}_2; \Gamma(\boldsymbol{r})t)=b(\boldsymbol{r}_1)^{-x}b(\boldsymbol{r}_2)^{-x}C(\boldsymbol{r}_1', \boldsymbol{r}_2'; b(\boldsymbol{r})^{-z}\Gamma(\boldsymbol{r})t). \tag{2.5}$$

It is also useful to record the corresponding result for the dynamic response function, whose Fourier transform $\tilde{G}(\boldsymbol{r}_1, \boldsymbol{r}_2; \omega\Gamma(\boldsymbol{r})^{-1})$ depends on Γ only through the indicated combination $\omega\Gamma^{-1}$, and is related to the Fourier transform of C by the fluctuation-dissipation theorem

$$\tilde{C}(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega) = (2/\omega) \operatorname{Im} \tilde{G}(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega). \tag{2.6}$$

The static limit ($\omega \to 0$) of $\tilde{G}$ gives the equilibrium correlation function $C(\boldsymbol{r}_1, \boldsymbol{r}_2; t_1 = t_2)$. Under a conformal transformation $\tilde{G}$ behaves according to

$$G(\boldsymbol{r}_1, \boldsymbol{r}_2; \Gamma(\boldsymbol{r})^{-1}\omega) = b(\boldsymbol{r}_1)^{-x} b(\boldsymbol{r}_2)^{-x} G(\boldsymbol{r}_1', \boldsymbol{r}_2'; b(\boldsymbol{r})^z \Gamma(\boldsymbol{r})^{-1}\omega). \tag{2.7}$$

2.1. *The logarithmic transformation*

In two dimensions, it is convenient to associate with each point $\boldsymbol{r}$ a complex number ζ. Any analytic function $w = f(\zeta)$ then corresponds to a conformal transformation, with $b(\boldsymbol{r}) = |f'(\zeta)|^{-1}$. Under such a transformation we have, from (2.5)

$$C(\zeta_1, \zeta_2; \Gamma(\zeta)t) = |f'(\zeta_1)|^x |f'(\zeta_2)|^x C(f(\zeta_1), f(\zeta_2); |f'(\zeta)|^z \Gamma(\zeta)t). \tag{2.8}$$

In general, the correlation functions on either side of this equation will be evaluated in different geometries. A particularly useful transformation is $w = \ln \zeta$, which maps the whole ζ plane into the strip $|\operatorname{Im} w| \leq \pi$, with periodic boundary conditions. The consequences of this for the static correlation function have already been studied by Cardy (1984). Putting $w = y + \mathrm{i}\theta$, and using the result that the static correlation function in the whole plane is simply $|\zeta_1 - \zeta_2|^{-2x}$, the correlation function in the strip is

$$C^{\mathrm{s}}(y_1, \theta_1; y_2, \theta_2) = [2\cosh(y_1 - y_2) - 2\cos(\theta_1 - \theta_2)]^{-x}. \tag{2.9}$$

For $y_1 - y_2 \to \infty$ this behaves like $\exp[-x(y_1 - y_2)]$, implying that the correlation length in the strip is $\xi = x^{-1}$. For a strip of width L, this generalises to $\xi(L) = L/2\pi x$. This result has been numerically verified for several isotropic two-dimensional models (Derrida and de Seze 1982, Nightingale and Blöte 1983). The main point is that the correlations in the strip are non-critical, as they must be for a quasi-one-dimensional system.

Now we can use (2.8) to express the dynamic correlation function in the plane (with a uniform $\Gamma = \Gamma_0$) in terms of the correlation function in the strip (with a non-uniform Γ):

$$C(r_1, \theta_1; r_2, \theta_2; \Gamma_0 t) = (r_1 r_2)^{-x} C^{\mathrm{s}}(\ln r_1, \theta_1; \ln r_2, \theta_2; \mathrm{e}^{-zy}\Gamma_0 t) \tag{2.10}$$

where a superscript s refers to the strip geometry. Similarly, the Fourier transformed response functions in the two geometries are related by

$$\tilde{G}(r_1, \theta_1; r_2, \theta_2; \omega\Gamma_0^{-1}) = (r_1 r_2)^{-x} \tilde{G}^{\mathrm{s}}(\ln r_1, \theta_1; \ln r_2, \theta_2; \omega\, \mathrm{e}^{zy}\Gamma_0^{-1}). \tag{2.11}$$

The critical dynamics in the plane is therefore related to the dynamics of a non-critical one-dimensional system with a non-uniform rate $\Gamma(y) = \mathrm{e}^{-zy}\Gamma_0$. Without loss of generality we may choose $r_2 = 1$, $\theta_2 = 0$. Since we are interested in the limit $r_1 \to \infty$, it is convenient to average over θ_1; that is, define

$$\tilde{G}_{\mathrm{av}}(r; \omega\Gamma_0^{-1}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\theta\, \tilde{G}(r, \theta; 1, 0; \omega\Gamma_0^{-1}). \tag{2.12}$$

Both $\tilde{G}_{\mathrm{av}}$ and $\tilde{G}$ have the same scaling limit and we shall henceforth ignore the distinction. The advantage is that the θ dependence has been averaged over on the right-hand side of (2.11), and since $\ln r \gg 1$, we are studying the truly one-dimensional properties of the strip.

3. Results for specific models

In this section we solve the problem of calculating the dynamic response function for a non-uniform one-dimensional system. Since we are interested only in the functional dependence of this quantity, and in distances much larger than the correlation length, it is permissible to apply mean-field or van Hove theory (Ma 1976). For a translationally invariant system, with no mode-mode coupling terms, this theory implies that the Fourier transform G_k of the response function satisfies

$$\partial G_k/\partial t = -\Gamma_k(k^2+\xi^{-2})G_k + \Gamma_k\delta(t) \tag{3.1}$$

where Γ_k is the relaxation rate for a mode with wavevector k, and ξ is the static correlation length. The cases of non-conserved order parameter (model A) and conserved order parameter (model B) correspond respectively to $\Gamma_k \to$ constant, and to $\Gamma_k \propto k^2$, as $k \propto 0$. For a non-uniform rate $\Gamma(y)$ this equation must be rewritten in coordinate space. We then find for model A that

$$\partial G^s(y, y_2)/\partial t = -\Gamma(y)[(-\partial^2/\partial y^2+\xi^{-2})G^s - \delta(y-y_2)\delta(t)] \tag{3.2}$$

while for model B

$$\partial G^s(y, y_2)/\partial t = (\partial/\partial y)\Gamma(y)(\partial/\partial y)[(-\partial^2/\partial y^2+\xi^{-2})G^s - \delta(y-y_2)\delta(t)] \tag{3.3}$$

where y is the coordinate measured along the strip. Note that if we Fourier transform (3.2) and (3.3) with respect to time and take the zero frequency limit, we obtain the correct mean-field equation satisfied by the static correlation function

$$(-\partial^2/\partial y^2+\xi^{-2})C^s(y, y_2) = \delta(y-y_2) \tag{3.4}$$

which, of course implies that $C^s(y, y_2) \propto \exp(-|y-y_2|/\xi)$. Comparing with the exact result (2.9) we see that we should choose $\xi = x^{-1}$, and that this approximation is valid only when $|y-y_2| \gg \xi$. The form of (3.3) is such that $\int_{-\infty}^{\infty} \dot{G}\, dy = 0$, corresponding to a conserved order parameter. In the limit $\xi \to 0$ it describes the diffusion of a particle in a non-uniform medium.

From these equations and (2.9) we may obtain in each case a differential equation obeyed by the response function in the plane, by choosing $\Gamma(y) = \Gamma_0\, e^{-zy}$.

Model A

In this case the differential equation is

$$(\partial/\partial t)r^x G(r, t) = -\Gamma_0 r^{-z}[-(r\,\partial/\partial r)^2 + x^2]r^x G + \Gamma_0\delta(r-1)\delta(t) \tag{3.5}$$

where we have put $y_2 = 0$. The peculiar form of the source term in (3.5) results from approximating G by G_{av}. Since we are interested in distances $r \gg 1$, the precise form of the source term is irrelevant, and we shall henceforth ignore it. It is convenient to

Laplace transform (3.5), defining

$$\bar{G}(r, s) = \int_0^\infty e^{-st} G(r, t)\, dt = \tilde{G}(r, is). \tag{3.6}$$

$\bar{G}$ then satisfies an ordinary differential equation:

$$[-(r\, d/dr)^2 + x^2] r^x \bar{G} = -(sr^{z+x}/\Gamma_0)\bar{G}. \tag{3.7}$$

From this it can be seen that $r^x\bar{G}$ depends on r through the combination sr^z only. Substituting $r^x\bar{G} = g$ and $s^{1/2}r^{z/2}\Gamma_0^{-1/2} = v$ the equation reduces to

$$v^2\, d^2g/dv^2 + v\, dg/dv - (4/z^2)(v^2 + x^2)g = 0 \tag{3.8}$$

which is the modified Bessel equation, with solutions $g = I_{2x/z}(2v/z)$, $K_{2x/z}(2v/z)$. The condition that $\bar{G}$ should be bounded as $r \to \infty$ picks out the second solution. As a result we find that

$$\bar{G}(r, s) = A(s) r^{-x} K_{2x/z}(s^{1/2} r^{z/2}) \tag{3.9}$$

where we have absorbed the irrelevant constant $(2/z)$ in the argument into a redefinition of r. The function $A(s)$ is fixed by the requirement that, as $s \to 0$, $\bar{G}$ should become the static correlation function r^{-2x}. This implies that, up to a constant, $A(s) = s^{x/z}$. The response function in real time may be recovered by inverse Laplace transform, with the remarkably simple result (Erdelyi *et al* 1954)

$$G(r, t) = t^{-2x/z-1} \exp(-r^z/t). \tag{3.10}$$

It is indeed straightforward to show that this satisfies the homogeneous form of (3.5), which can in fact be solved by quadratures.

Both forms (3.9) and (3.10) are the unique solution for the critical dynamic response function up to a non-universal rescaling of r and t. They are, of course, consistent with dynamic scaling. The result (3.10) is very appealing, being almost the simplest modification of the van Hove theory result $t^{-1}\exp(-r^2/t)$ of which one could conceive. The non-analyticity of the asymptotic form (3.10) at $r = 0$ is not inconsistent with the presumed regularity of $G(r, t)$ at $r = 0$ for finite t. In our calculation we had to assume that r is large.

The dynamic structure factor, as measured in a scattering experiment, is related to the Hankel transform of $\bar{G}(r, s)$:

$$S(k, \omega) = \frac{2}{\omega} \operatorname{Im} \int_0^\infty dr\, r J_0(kr) \bar{G}(r, -i\omega). \tag{3.11}$$

We have been able to evaluate this analytically only in the case $z = 2$ (which is a good approximation for the Ising model). We find (Erdelyi *et al* 1954)

$$S(k, \omega) = (2/\omega) \operatorname{Im}(-i\omega + k^2)^{-1+\eta/2} \tag{3.12}$$

$$= (2/\omega)(\omega^2 + k^4)^{-1/2+\eta/4} \sin[(1 - \eta/2)\tan^{-1}(\omega/k^2)]. \tag{3.13}$$

Model B

The differential equation satisfied by the response function is now of fourth order

$$r^x\, \partial G/\partial t = \Gamma_0 (r\, d/dr) r^{-z} (r\, d/dr)[-(r\, d/dr)^2 + x^2] r^x G. \tag{3.14}$$

For simplicity, we assume from the beginning the scaling form $G(r,t)=(\Gamma_0 t)^{-2x/z-1}\psi(u)$, where $u=r^z/\Gamma_0 t$. After some algebra, we find that ψ satisfies

$$(\delta+2x/z)[\delta(\delta+x/z-1)(\delta+x/z)\psi-(u/z^4)\psi]=0 \tag{3.15}$$

where $\delta=u\,\mathrm{d}/\mathrm{d}u$. This is a generalised hypergeometric equation (Erdelyi *et al* 1953). These are four independent solutions which behave like u^α as $u\to 0$, where $\alpha=0$, $-2x/z$, $1-x/z$, $-x/z$ respectively. Since G must be finite at $u=0$ ($r=0$), only the solutions with $\alpha=0$, $1-x/z$ are allowed. This means that in fact ψ satisfies the third-order equation

$$\delta(\delta+x/z-1)(\delta+x/z)\psi-(u/z^4)\psi=0. \tag{3.16}$$

The solution which is regular at $u=0$ is

$${}_0F_2(x/z,1+x/z;u/z^4). \tag{3.17}$$

The other solution which is finite at $u=0$ may be found by letting $\psi=u^{1-x/z}\bar\psi$; $\bar\psi$ then satisfies a similar third-order hypergeometric equation, and we choose the solution regular at $u=0$. The second solution of (3.16), finite at $u=0$, is then

$$u^{1-x/z}{}_0F_2(2,2-x/z;u/z^4). \tag{3.18}$$

To determine the correct linear combination of these two solutions it is necessary to study the behaviour as $u\to\infty$. From the differential equation we see that the asymptotic behaviour is of the form

$$\psi\sim u^\alpha\exp(\lambda\mu^\beta) \tag{3.19}$$

and, on substituting this into equation (3.16) we find $\beta=\frac{1}{3}$ and $(\lambda/3)^3=z^{-4}$, so that a typical solution will blow up like $\exp[3(u/z^4)^{1/3}]$. We must therefore choose the correct linear combination of (3.17) and (3.18) so that this divergence is cancelled. We require the asymptotic behaviour of the ${}_0F_2$ function. Since this does not appear in the standard literature we sketch the details of this calculation. Define

$$f(x)={}_0F_2(\rho_1,\rho_2,(x/3)^3). \tag{3.20}$$

The Laplace transform is (Erdelyi *et al* 1954)

$$g(p)=\int_0^\infty \mathrm{e}^{-px}f(x)\,\mathrm{d}x=p^{-1}{}_3F_2(\tfrac{1}{2},\tfrac{2}{3},1;\rho_1,\rho_2;p^{-3})$$

$$=p^{-1}\sum_{n=0}^\infty\frac{(\frac{1}{3})_n(\frac{2}{3})_n}{(\rho_1)_n(\rho_2)_n}p^{-3n}. \tag{3.21}$$

This series converges for $|p|>1$, and the singularities on the circle of convergence are at $p=1$, $\mathrm{e}^{2\pi\mathrm{i}/3}$, $\mathrm{e}^{-2\pi\mathrm{i}/3}$. The nature of these singularities follows from the ratio test: denoting the series by $\Sigma_n b_n p^{-3n}$, we find that as $n\to\infty$

$$\frac{b_{n+1}}{b_n}\sim 1+\frac{1-\rho_1-\rho_2}{n}+\mathrm{O}(n^{-2}) \tag{3.22}$$

which means that the singularity is of the form $(1-p^{-3})^\alpha$ with $\alpha=\rho_1+\rho_2-2$. Taking the inverse Laplace transform,

$$f(x)\sim 3^{\rho_1+\rho_2-2}\frac{\Gamma(\rho_1)\Gamma(\rho_2)}{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})}x^{1-\rho_1-\rho_2}\,\mathrm{e}^x. \tag{3.23}$$

It is now straightforward to find the correct linear combination of (3.17) and (3.18). The final result for the scaling function is

$$\psi(u)=\Gamma(2-x/z)_0F_2(x/z,1+x/z;u/z^4)$$
$$-(u/z^4)^{1-x/z}\Gamma(x/z)\Gamma(1+x/z)_0F_2(2,2-x/z;u/z^4). \tag{3.24}$$

This result, although exact, is not particularly transparent. The main feature of $\psi(u)$ is its asymptotic behaviour as $u\to\infty$,

$$\psi(u)\sim\text{constant}\times u^{-2x/3z}\,e^{-u^{1/3}}\cos(\sqrt{3}u^{1/3}) \tag{3.25}$$

where some numerical constants have been absorbed into the (non-universal) normalisation of $u=r^z/\Gamma_0 t$. The oscillations in this function are characteristic of a system with a conserved order parameter. On setting $z=4$, $x=0$, we obtain the prediction of van Hove theory. In principle, the formula (3.25) can be Fourier transformed to obtain the dynamic structure factor, but we have no analytic results.

4. Summary and discussion

We have argued that the logarithmic conformal transformation maps the critical dynamics of a two-dimensional system onto those of a quasi-one-dimensional strip with a non-uniform rate. Solving the latter problem, we have derived the first exact results for the dynamic scaling function at criticality in two dimensions. Our main results are (3.10) for the response function with a non-conserved order parameter, and (3.25) in the conserved case. In principle the analysis can be extended to more complicated types of dynamics. Unfortunately it does not predict the value of the dynamic exponent z.

The result (3.10) has an application to directed lattice animals in three dimensions. It has been argued (Cardy 1982, Breuer and Janssen 1982) that this problem is equivalent to the model A dynamics of the Yang–Lee edge singularity in two dimensions. In this equivalence, the response function $G(r_\parallel, r_\perp)$ gives the density of monomers in a large directed animal rooted at the origin.

Finally, we suggest that the logarithmic conformal transformation may give a useful qualitative picture of a two-dimensional system at its critical point. Recall that the correlation length in the strip is equal to x. In a quasi-one-dimensional system such a finite correlation length results from a finite density x^{-1} of domain walls crossing the whole strip (figure 1(*a*)). Under conformal transformation, this array of domain walls maps into those shown in figure 1(*b*). This is just the scale-invariant picture of 'droplets within droplets' espoused by Fisher (1967) and considered quantitatively by Bruce and Wallace (1983) in $1+\varepsilon$ dimensions. Note that the only domain walls which contribute to the correlation function between points r_1 and r_2 are those for which (in some average sense) these points lie on opposite sides. The remaining droplets, required by translational invariance in the two-dimensional picture, play no role. It would be interesting to develop these qualitative ideas further. In particular, the dynamics of model A in one dimension can be understood in terms of a simple diffusive behaviour of the domain walls. The results in this paper, then, should give information on the diffusion of domain walls in two dimensions, where curvature also plays an important role.

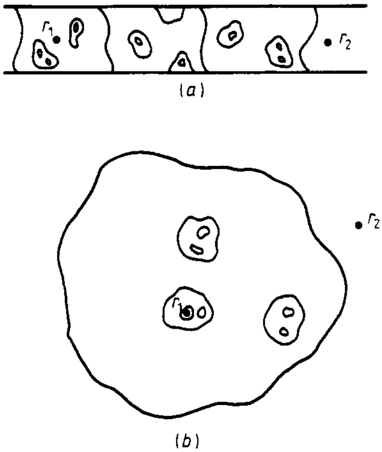

Figure 1. (*a*) Typical configuration of domain walls in the strip. The picture is supposed to be scale invariant on length scales smaller than the strip width. On larger scales there is a finite density x of walls which cross the strip. (*b*) The conformally transformed picture. The regular array of domain walls in the strip corresponds to a scale-invariant distribution in the plane. However, only domain walls passing between r_1 and r_2 contribute to the correlation function of spins at these points.

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